THE AFFINE IMAGE OF A CONVEX BODY OF CONSTANT BREADTH

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ABSTRACT

A convex body is said to have constant diagonal if and only if the main diagonal of the circumscribed boxes has constant length. It is shown that an n-dimensional convex body, $n \geqq 3$, is the affine image of a body of constant breadth if and only if it has constant diagonal. Affine images of bodies of constant breadth are also characterized by the property that the orthogonal projection on each hyperplane is the affine image of a body of constant breadth in that hyperplane.

A convex body in n-dimensional Euclidean space E_n is a compact, convex subset with non-empty interior. A convex body has "constant breadth" if the distance between parallel supporting hyperplanes is constant. A set K' is the "affine image" of K if there exists an affine transformation $f: E_n \to E_n$ such that $f(K) = K'$. A rectangular parallelopiped will be referred to as a "box". A convex body has "constant diagonal" if the main diagonal of the circumscribed boxes has constant length.

In this paper we are concerned with characterizing affine images of bodies of constant breadth, a problem raised by S.K. Stein. Our main theorem is,

THEOREM 1. *In E_n, $n \geqq 3$, a convex body is the affine image of a body of constant breadth if and only if K has constant diagonal.*

We shall also prove a related theorem,

THEOREM 2. *In E_n, $n \geqq 3$, a convex body is the affine image of a body of constant breadth if and only if its orthogonal projection on each hyperplane is the affine image of a body of constant breadth in that hyperplane.*

The proofs of these theorems depend on the following lemmas.

LEMMA 1. *In E_n, $n \geqq 2$, a convex body is the affine image of a body of constant breadth if and only if $K + (-K)$ is an ellipsoid.*

Proof. This follows immediately from the observation that K is a body of constant breadth if and only if $K + (-K)$ is spherical.

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LEMMA 2. *In E_n, $n \geqq 3$ a convex body is an ellipsoid if and only if its orthogonal projection on, each hyperplane is an ellipsoid in that hyperplane.*

Proof. If K is an ellipsoid, then all its orthogonal projections are ellipsoids. The following proof of the converse is an adaptation of the proof given by Süss [3] for the case of E_3. Now suppose each orthogonal projection of K is an ellipsoid. Then each supporting hyperplane of K intersects K in just one point. Denoting the points of E_n by $x = (x_1, \cdots, x_n)$, assume that the segment joining $a = (0, \cdots, 0, 1)$ to $a' = (0, \cdots, 0, -1)$ is a maximum diameter of K. Then the hyperplanes $x_n = 1$ and $x_n = -1$ are supporting hyperplanes of K. Let K^* be the orthogonal projection of K on the hyperplane $x_1 = 0$. Then K^* is an ellipsoid in $x_1 = 0$. aa' is one axis of K^*, since the intersections of $x_n = 1$ and $x_n = -1$ with $x_1 = 0$ are supporting $(n-2)$-planes of K^*, orthogonal to aa' and passing through a and a' respectively. Let H be a supporting hyperplane of K which is orthogonal to $x_1 = 0$. Then H intersects $x_1 = 0$ in H^*, where H^* is a supporting $(n-2)$-plane of K^*. If, in particular, H is also chosen parallel to aa', then H^* is parallel to aa', and hence $H^* \cap K^*$ lies in $x_n = 0$. It follows that $H \cap K$ lies in $x_n = 0$. But this argument could have been applied to any supporting hyperplane parallel to aa'; hence, any supporting hyperplane parallel to aa' intersects K in a point lying in $x_n = 0$. From this it follows that the intersection of K with $x_n = 0$ is identical with its orthogonal projection on $x_n = 0$, which is an ellipsoid K^{**}. It is easy to show that K^{**} must be centered at the origin. Now let $f: E_n \to E_n$ be an affinity which keeps aa' fixed and maps K^{**} onto a sphere S in $x_n = 0$ centered at the origin. Then $f(K)$ intersects $x_n = 0$ in the sphere S. Each diameter of S is orthogonal to the supporting hyperplanes of $f(K)$ through its endpoints. Also, $x_n = 1$ and $x_n = -1$ are supporting hyperplanes of $f(K)$, orthogonal to aa' and passing through a and a' respectively. Finally, all the orthogonal projections of $f(K)$ are ellipsoids (here one needs to use the fact that not only the orthogonal projections, but all projections, of K are ellipsoids). In the argument above, the only property of aa' we actually used was that the supporting hyperplanes through a and a' were orthogonal to aa'. From this it followed that the hyperplane through the origin orthogonal to aa' intersected K in an ellipsoid. Thus if bb' is any diameter of S, the same arguments can be applied to show that the hyperplane through the origin orthogonal to bb' intersects $f(K)$ in an ellipsoid; moreover, this ellipsoid has aa' as an axis. It follows that every 2-plane containing aa' intersects K in an ellipse having aa' as one axis and a diameter of S as the other. Thus $f(K)$ is an ellipsoid of revolution, and K is an ellipsoid. This completes the proof.

LEMMA 3. *In E_n, $n \geqq 3$, a convex body is an ellipsoid if and only if all its circumscribed boxes have their vertices on a fixed sphere.*

Proof. The sufficiency of the condition is proved, for $n = 3$, in [1]. We proceed to the general case by induction. Suppose K is a convex body in E_n, $n > 3$, all of whose circumscribed boxes have their vertices on a fixed sphere S, and assume we know the lemma for E_k, $3 \leqq k < n$. Let H be any supporting hyperplane of K, and let K^* be the orthogonal projection of K on H. Then any box B^* in H circumscribed about K^* is a face of a box B circumscribed about K. The vertices of B lie on S; hence, the vertices of B^* lie on the sphere $S \cap H$. Thus K^* is an ellipsoid in H. It follows that the orthogonal projection of K on any hyperplane is an ellipsoid, so by Lemma 2, K is an ellipsoid. The result follows, for all n, by induction. The converse, namely that all circumscribed boxes of an ellipsoid have their vertices on a fixed sphere, is a simple matter of analytic geometry. This completes the proof.

Proof of Theorem 1. 1. Let S^{n-1} be the unit sphere centered at the origin in E_n. A "direction" in E_n is a point $u \in S^{n-1}$. Now suppose K is the affine image of a body of constant breadth, so $K + (-K)$ is an ellipsoid E centered at the origin. Let $p(u)$ be the support function of E measured from the origin, and let $b(u)$ be the breadth function (distance between parallel supporting hyperplanes orthogonal to direction u) of K. Then $p(u) = b(u)$ for all directions u. But if $u_1, u_2, \cdots, u_n$ are any n mutually orthogonal directions, then $\sum_{i=1}^{n}[p(u_i)]^2$ is constant, by Lemma 3. Hence, $\sum_{i=1}^{n}[b(u_i)]^2$ is constant, which is precisely the condition that K have constant diagonal. Conversely, if K has constant diagonal, then $\sum_{i=1}^{n}[b(u_i)]^2$ is constant, so $\sum_{i=1}^{n}[p(u_i))]^2$ is constant, where $p(u)$ is the support function of $K + (-K)$. Thus all the boxes circumscribed about $K + (-K)$ have their vertices on a fixed sphere. By Lemma 3, $K + (-K)$ is an ellipsoid, so K is the affine image of a body of constant breadth. This completes the proof of the theorem.

Proof of Theorem 2. For each direction u let E_u be the hyperplane through the origin orthogonal to u. Let K_u be the orthogonal projection of K on E_u. If K is the affine image of a body of constant breath, then $K + (-K)$ is an ellipsoid. Hence, $[K + (-K)]_u = K_u + (-K_u)$ is an ellipsoid in E_u, so K_u is the affine image of a body of constant breadth in E_u. Conversely, suppose K_u is the affine image of a body of constant breadth in E_u, for each u. Then $[K + (-K)]_u = K_u + (-K_u)$ is an ellipsoid in E_u for each u. By Lemma 2, $K + (-K)$ is an ellipsoid, so K is the affine image of a body of constant breadth. This completes the proof.

REMARK. An interesting characterization of affine images of curves of constant breadth in E_2 is to be desired. While Theorem 1 is true in one direction in the plane case, viz. an affine image of a curve of constant breadth has constant diagonal, the converse is false. Blaschke, in [2], gives examples of centrally symmetric convex curves with constant diagonal which are not ellipses. Such a curve could not be the affine image of a curve of constant breadth, since the only centrally symmetric curve of constant breadth is the circle.

REFERENCES

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